Noise Field of a Rotating Propeller in Forward Flight

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The propeller blade is replaced by a source and doublet distribution referring to the acceleration potential. In the linearized approach used here, these distributions are situated at the helicoidal surface described by the propeller. The doublet distribution leads to a lifting surface problem, whereas the source distribution corresponds to a thickness problem for a usually nonsymmetrical blade. These problems can be solved independently. Results have been obtained for the pressure variations and for the noise level in certain points due to each of the two cases. It appears that the noise levels are of the same order of magnitude for the two cases. The essential step in the solution is the calculation of the field due to a source moving along a helicoidal line. This field has been obtained in the form of a parameter representation. The calculation of the field due to a doublet moving along a helicoidal line also has been performed.

Nomenclature

c = speed of sound
$\bar{e}$ = doublet strength
f^+,f^- = functions giving the geometry of suction side and pressure side of the propeller blade
n = number of blades
p = total pressure
p_m = pressure amplitude of mth harmonic
r,r_0 = radial coordinate
s_0 = $[x^2 + \beta^2(r^2 + r_0^2 - 2rr_0 \cos\alpha)]^{1/2}$
t = time
u,v,w = perturbation velocities in the direction of the coordinate axes
x,y,z = rectangular coordinate system at rest with respect to the propeller
E = VU
M = Mach number
Q = source strength
T = total thrust of propeller
U = uniform velocity of propeller
V = volume of propeller blade
α = parameter
β = $(1 - M^2)^{1/2}$
ϵ = small quantity
ρ = density
σ = $s_0 + (\omega/c)rr_0 \sin\alpha$
φ = angular coordinate
χ = velocity potential
ψ = acceleration potential
ω = angular velocity of propeller around x axis

1. Introduction

THE problem of the sound field due to a rotating propeller has been investigated theoretically by Gutin[1] for small forward speeds. His method later was extended and generalized by Garrick and Watkins.[2] In this method, the action of the propeller is replaced by fixed periodic forces at the propeller disk. These forces are obtained as a result of acceleration sources and doublets in the disk which move uniformly along straight lines, and the strength of which varies. A harmonic analysis is applied so that, in fact, a number of harmonically varying singularities need to be considered. A somewhat similar investigation on the noise due to thickness has been performed by Arnoldi.[3]

A slightly different approach was used by Zandbergen and van der Walle.[4] There a discussion is given about the plausibility of the various assumptions that usually are made. The uncertainty of the results obtained was one of the main reasons for undertaking the present investigation.

In this paper, a solution is obtained which contains as the only approximations the assumption of linearized flow and the neglect of viscosity. With the aid of a Lorentz-Galileo transformation, it is possible to calculate the field due to a singularity moving along a helicoidal line. This yields also the variation of pressure as a function of time during a propeller revolution at any point in space moving with the airplane. A harmonic analysis needs only be applied to this result if the sound intensity expressed in decibels is required. Calculations have been performed for the noise caused by thrust and torque and for the noise caused by thickness. Although these calculations refer to one doublet for each case, they can be extended to a larger number of singularities, giving a better representation of the real distributions.

2. Boundary Value Problem

Let x',y',z' be a rectangular coordinate system (Fig. 1) at rest with respect to the undisturbed medium. The propeller axis coincides with the x' axis while the airplane is moving with constant velocity U in the direction of the negative x' axis. The equation governing the propagation of small disturbances is the wave equation

$$\frac{\partial^2\psi}{\partial x'^2} + \frac{\partial^2\psi}{\partial y'^2} + \frac{\partial^2\psi}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2} = 0$$

which holds for both the velocity and the acceleration potential and where c denotes the speed of sound.

Introducing a second coordinate system x,y,z that is moving with the airplane by the transformation

$$x = x' + Ut' \qquad y = y' \qquad z = z' \qquad t = t'$$

the differential equation becomes

$$(1 - M^2)\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - 2\frac{M}{c}\frac{\partial^2\psi}{\partial x\partial t} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

where $M = U/c$ is the Mach number. In this coordinate system, the speed of the undisturbed air is U in the direction of the positive x axis.

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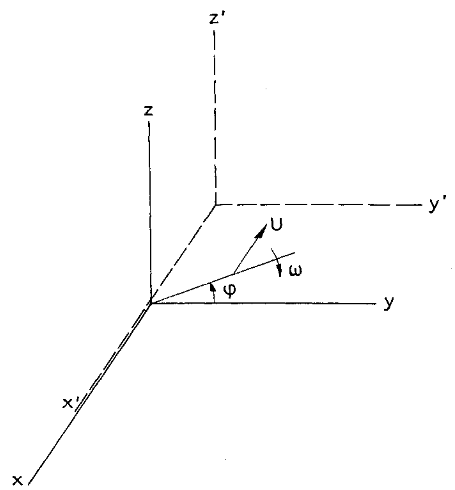

Fig. 1 Coordinate system x', y', z' is at rest; system x, y, z moves with airplane velocity U

In the case of the velocity potential, the boundary condition follows from the requirement that at the propeller blade the velocity component of the air normal to the blade should be equal to that of the blade. The solution is determined only uniquely if, moreover, a condition (usually the Kutta condition) is introduced for determining the circulation.

The boundary condition already has been formulated by Sparenberg.[5] First consider a one-bladed propeller, but the eventual extension of the theory to a multibladed propeller offers no difficulties. The blade rotates with an angular velocity ω around the x axis (Fig. 1). Representing the blade by a single line for a moment and introducing x, r, and φ as a system of cylindrical coordinates, one has

$$\varphi = -\omega t \qquad x' = -Ut$$

Hence this line describes the helicoidal surface $\varphi - (\omega/U)x' = 0$ (Fig. 2). In the x,y,z coordinates, this becomes a rotating helicoidal surface

$$H(x,r,\varphi,t) \equiv \varphi - (\omega/U)x + \omega t = 0 \tag{2.1}$$

An infinitely thin blade exactly coinciding with a part of this surface would not exert any action on the fluid. The real blade has the equation

$$F(x,r,\varphi,t) \equiv \varphi - (\omega/U)x + \omega t + \epsilon f(x,r) = 0 \tag{2.2}$$

where ϵ is a small constant and $f(x,r)$ is a two-valued function, one value f^+ denoting the suction side and the other value f^- denoting the pressure side. Because of the smallness of ϵ, the blade causes small disturbances in the medium. The domain G of x and r corresponding to the blade is

$$x_l(r) \leqslant x \leqslant x_t(r) \qquad r_i \leqslant r \leqslant r_o$$

where $x_l(r)$ and $x_t(r)$ denote the leading and the trailing edges of the blade, and r_i and r_o are the inner and the outer radii of the blade.

Let the components of the velocity of the fluid be $U + u$, v,w in the x, r, and φ directions, respectively. The direction of the normal to the blade is identical to the direction of grad F, which means that $C(\partial F/\partial x)$, $C(\partial F/\partial r)$, and $(C/r)(\partial F/\partial \varphi)$, with

$$C = \frac{1}{[(\partial F/\partial x)^2 + (\partial F/\partial r)^2 + (1/r^2)(\partial F/\partial \varphi)^2]^{1/2}} \tag{2.3}$$

are direction cosines of the normal. Hence the velocity component of the air normal to the blade is

$$C\left\{(U + u)\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial r} + \frac{w}{r}\frac{\partial F}{\partial \varphi}\right\}$$

whereas the velocity component of the blade itself in this direction is equal to $-\omega C(\partial F/\partial \varphi)$.

From the condition that these two velocities should be equal, it follows after substitution of Eq. (2.2) and neglect of terms small of second order in ϵ (u, v, and w are assumed to be small of first order in ϵ) that

$$\epsilon U(\partial f/\partial x) = (\omega/U)u - (w/r) \tag{2.4}$$

It is in agreement with the neglect of ϵ^2 to apply this condition at the surface $H = 0$ instead of at the surface $F = 0$.

In the present case, it is of advantage to formulate the problem in terms of the acceleration potential. The boundary condition then follows from the fact that the normal component of the acceleration at the blade is determined by the blade geometry. Together with the requirement that the acceleration potential should be regular at infinity, this completely determines the boundary value problem.

The acceleration component normal to the blade becomes[5]

$$a_n = C\left\{\left(U\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right)\frac{\partial F}{\partial x} + \left(U\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t}\right)\frac{\partial F}{\partial r} + \left(U\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t}\right)\frac{\partial F}{r\partial \varphi}\right\}$$

Substituting Eq. (2.2) for F and making use of Eq. (2.4), it is found that

$$a_n = \epsilon\, C\, U^2(\partial^2 f/\partial x^2) \tag{2.5}$$

with $C = rU/(\omega^2 r^2 + U^2)^{1/2}$. This condition has to be applied at both sides of the surface $H = 0$.

3. Separation of the Total Problem into a Lifting and a Thickness Problem

Since the function f assumes two different values at the two sides of the real blade, the normal acceleration, in general, also will be different. Moreover, there will be a difference in pressure, and hence also in acceleration potential, between the two sides. This means that in the linearized approximation there will be a discontinuity at the surface $H = 0$ in both the potential and its normal derivative.

It is well known from potential theory[6] that a discontinuity in the potential can be represented by a doublet distribution

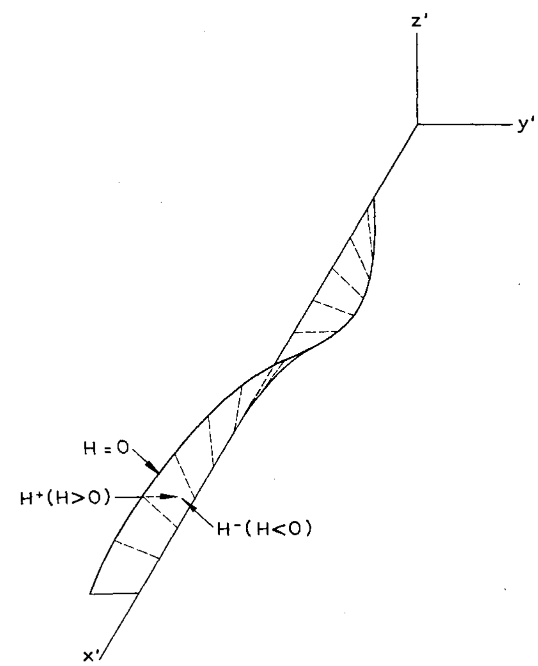

Fig. 2 Helicoidal surface described by a rotating line

Table 1 Separation into thickness and lifting problems

	Suction side	Pressure side	Difference
Thickness (sources)			
Discontinuous	$\frac{1}{2}\frac{\partial^2(f^+ - f^-)}{\partial x^2}$	$-\frac{1}{2}\frac{\partial^2(f^+ - f^-)}{\partial x^2}$	$\frac{\partial^2(f^+ - f^-)}{\partial x^2}$
Continuous	$\frac{\partial^2 f_1}{\partial x^2}$	$\frac{\partial^2 f_1}{\partial x^2}$	0
Lifting (doublet)	$\frac{\partial^2}{\partial x^2}\left(\frac{f^+ + f^-}{2} - f_1\right)$	$\frac{\partial^2}{\partial x^2}\left(\frac{f^+ + f^-}{2} - f_1\right)$	0
Total	$\frac{\partial^2 f^+}{\partial x^2}$	$\frac{\partial^2 f^-}{\partial x^2}$	$\frac{\partial^2(f^+ - f^-)}{\partial x^2}$

and a discontinuity in the normal derivative by a source distribution, both in the surface where the discontinuities occur. The doublet strength per unit area then is equal to the local potential jump, and the source strength per unit area is equal to the local discontinuity in normal derivative. These results hold not only when the discontinuity surface is at rest but also when it moves and when the differential equation is a form of the wave equation instead of Laplace's equation. The second half of this statement follows from the fact that near a singularity the solutions of the wave equation and of the Laplace equation behave identically. The first half can be verified by remarking that the near field, due to a source that at a certain moment is moving with speed U, is given by[7]

$$-\frac{1}{4\pi}\frac{Q}{[x^2 + \beta^2(y^2 + z^2)]^{1/2}}$$

where $\beta = [1 - (U^2/c^2)]^{1/2}$.

The discontinuity in the normal derivative is, according to Eq. (2.5), equal to $\epsilon\, C\, U^2(\partial^2/\partial x^2)(f^+ - f^-)$, which, hence, also is equal to the strength of the source distribution. Since the surface $H = 0$, where the distribution is situated, is not a plane surface, the sources also will induce a continuous component of the normal derivative. This component can be obtained by integration of a function containing the known source distribution as factor. The integration is over the part of the surface $H = 0$ corresponding to the blade and, if more blades are considered, also over the other blades. The resulting continuous component of the normal derivative will be written as $\epsilon\, C\, U^2(\partial^2 f_1/\partial x^2)$, where f_1 is determined uniquely by requiring that

$$f_1(x_l,r) = f_1(x_t,r) = 0$$

Since the total value of the normal derivative is known for both sides of the blade, there remains a continuous part (see Table 1) equal to

$$\epsilon C U^2 \frac{\partial^2}{\partial x^2}\left(\frac{f^+ + f^-}{2} - f_1\right)$$

This is caused by the doublet distribution. Hence the lifting problem can be represented by an infinitely thin blade of which

$$\epsilon\{[(f^+ + f^-)/2] - f_1\}$$

gives the distance from the surface $H = 0$ in its dependence of x and r. The thickness problem corresponds to a nonsymmetrical blade for which the distances of suction and pressure sides from the surface $H = 0$ are

$$\epsilon\left(\frac{f^+ - f^-}{2} + f_1\right) \qquad \epsilon\left(-\frac{f^+ - f^-}{2} + f_1\right)$$

respectively. This unsymmetrical blade, when moving along the surface $H = 0$, will have equal pressures at corresponding points of its two sides.

The values in Table 1 give, after multiplication by $\epsilon C U^2$, the normal acceleration at the blade.

It is seen that the solution ψ of the complete problem is obtained in this linearized theory as the sum of the solutions for the thickness and the lifting problems.

4. Potential Due to a Source Moving along a Helicoidal Line

Consider the xyz coordinate system moving with the airplane and take the source at a distance r_0 from the x axis about which it rotates with an angular velocity (see Fig. 1). At the moment $t = 0$, the source is at the y axis. Since in this paper only the potential due to a constant source is needed, the source is taken as constant, but the final formula can be extended to the case of a variable source strength.

The differential equation is

$$(1 - M^2)\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - 2\frac{M}{c}\frac{\partial^2\psi}{\partial x\partial t} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = Q\,\delta(x)\delta(y - r_0\cos\omega t)\delta(z + r_0\sin\omega t)$$

This shows that there is a concentrated source of strength Q at the point $x = 0$, $y = r_0\cos\omega t$, $z_0 = -r_0\sin\omega t$, whereas the source distribution is zero everywhere else.

In cylinder coordinates, the equation becomes

$$(1 - M^2)\frac{\partial^2\psi}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2} - 2\frac{M}{c}\frac{\partial^2\psi}{\partial x\partial t} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = Q\,\delta(x)\frac{\delta(r - r_0)}{r}\delta(\varphi + \omega t) \quad (4.1)$$

The factor r in the denominator of the right-hand side determines that, after integration over the whole space, the total source strength is again Q, since

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\delta(x)\frac{\delta(r - r_0)}{r}\delta(\varphi + \omega t)\,dx\,dr\,rd\varphi = 1$$

Assume now that the velocity U is subsonic. The transformation is introduced:

$$X = x \qquad R = \beta r \qquad \Phi = \varphi \qquad T = \beta^2 ct + Mx$$

where $\beta = (1 - M^2)^{1/2}$.

This transforms the equation for the acceleration potential into a form of the wave equation, viz.,

$$\frac{\partial^2\psi}{\partial X^2} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\Phi^2} - \frac{\partial^2\psi}{\partial T^2} = \frac{Q}{\beta}\,\delta(X)\,\frac{\delta[(R - R_0)/\beta]}{R}\delta\left(\Phi + \omega\frac{T - MX}{\beta^2 c}\right) \quad (4.2)$$

In order to solve this equation, first turn to a more familiar form of the wave equation, viz.,

$$\frac{\partial^2\psi}{\partial x'^2} + \frac{\partial^2\psi}{\partial y'^2} + \frac{\partial^2\psi}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2} = q(x',y',z',t')$$

of which the solution is known to be[8]

$$\psi(x',y',z',t') = -\frac{1}{4\pi}\iiint_{-\infty}^{\infty}\frac{q[\xi',\eta',\zeta',t' - (R/c)]}{R}\,d\xi' d\eta' d\zeta'$$

where

$$R = [(x' - \xi')^2 + (y' - \eta')^2 + (z' - \zeta')^2]^{1/2}$$

Hence, the solution of Eq. (4.2) becomes

$$\psi(X,R,\Phi,T) = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{Q}{\beta S'}\,\delta(\xi)\times \frac{\delta\,[(R_1 - R_0)/\beta]}{R_1}\,\delta\left(\Phi_1 + \omega\frac{T - S' - M\xi}{\beta^2 c}\right)d\xi\; dR_1\; R_1 d\Phi_1 \quad (4.3)$$

where

$$S' = [(X - \xi)^2 + R^2 + R_1^2 - 2RR_1\cos(\Phi - \Phi_1)]^{1/2}$$

denotes the distance between the points (X,R,Φ) and (ξ,R_1,Φ_1).

The integrations toward ξ and R_1 can be performed easily, remembering that

$$\int_{-\infty}^{\infty}\delta(ky)dy = \frac{1}{|k|}$$

The result for ψ then is

$$\psi(X,R,\Phi,T) = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{Q}{S}\,\delta\left(\Phi_1 + \omega\frac{T - S}{\beta^2 c}\right)d\Phi_1$$

where

$$S = [X^2 + R^2 + R_0^2 - 2RR_0\cos(\Phi - \Phi_1)]^{1/2} \quad (4.4)$$

For the further reduction, one puts

$$\Phi_1 + \omega[(T - S)/\beta^2 c] = \mu \quad (4.5)$$

and consider μ as a new integration variable. Then

$$\psi(X,R,\Phi,T) = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{Q\delta(\mu)}{S + (\omega/\beta^2 c)RR_0\sin(\Phi - \Phi_1)}\,d\mu \quad (4.6)$$

Hence, the value of the denominator for $\mu = 0$ has to be evaluated. The quantities S and Φ_1 both depend upon μ, Denoting their values for $\mu = 0$ by S_0 and Φ_0, respectively, one has from Eqs. (4.4) and (4.5)

$$S_0 = (X^2 + R^2 + R_0^2 - 2\,RR_0\cos\{\Phi - \Phi_0\})^{1/2}$$
$$\Phi_0 + \omega[(T - S_0)/\beta^2 c] = 0 \quad (4.7)$$

These are two transcendental equations for S_0 and Φ_0. However, it appears possible to obtain a closed expression for ψ in the form of a parameter representation.

Putting

$$\Phi = \Phi_0 + \alpha \quad (4.8)$$

the solution (4.6) becomes

$$\psi(X,R,\Phi,T) = -\frac{Q}{4\pi[S_0 + (\omega/\beta^2 c)RR_0\sin\alpha]} \quad (4.9)$$

where

$$S_0 = (X^2 + R^2 + R_0^2 - 2\,RR_0\cos\alpha)^{1/2} \quad (4.10)$$

What remains to be done is to express Φ as function of α. From Eq. (4.8), it is seen that Φ_0 should be expressed as function of α. This can be done with the aid of the second Eq. (4.7) and Eq. (4.10). The result is

$$\Phi = \alpha - (\omega/\beta^2 c)[T - (X^2 + R^2 + R_0^2 - 2RR_0\cos\alpha)^{1/2}] \quad (4.11)$$

Equations (4.9–4.11) give the solution for the potential due to a constant source moving with constant speed along a helicoidal line. Returning to the original cylinder coordinates x,r,φ,t, one obtains

$$\psi(x,r,\varphi,t) = -\frac{Q}{4\pi[s_0 + (\omega/c)rr_0\sin\alpha]} \quad (4.12)$$

$$s_0 = [x^2 + \beta^2(r^2 + r_0^2 - 2r\,r_0\cos\alpha)]^{1/2} \quad (4.13)$$

$$\varphi + \omega t = \alpha - (\omega/\beta^2 c)(Mx - s_0) \quad (4.14)$$

Physically, the potential is determined by a disturbance that has been emitted by the source at some previous instant (retarded potential). It follows from the foregoing analysis that $\varphi_1 = \Phi_1$ denotes the position of the source. Since in Eq. (4.6) only $\mu = 0$ contributes to the potential, the position of the source from which a signal is received is equal to Φ_0. The position of the source for $t = 0$ is $\omega = 0$. The signal then was emitted at time $-(\Phi_0/\omega)$. By aid of Eqs. (4.8) and (4.14), it follows that this is equal to

$$t - [(s_0 - Mx)/\beta^2 c] \quad (4.15)$$

Hence, if the source strength is variable and if the potential ψ is considered at the moment t, the value of Q in Eq. (4.12) should be taken at the time given by (4.15). This same result also follows if in the derivations leading to (4.12) Q had been taken from the beginning as a function of time.

In order to obtain the variation of ψ in a fixed point (x,r,φ) as a function of the time t during one revolution of the propeller, proceed as follows. Let α vary from 0 to 2π. Then find from Eqs. (4.12–4.14) ψ and $\varphi + \omega t$ as functions of α. If $\varphi + \omega t$ is a monotonically increasing function of α, it is trivial to obtain ψ as a function of $\varphi + \omega t$, which, when φ is kept constant, also gives ψ as a function of t. If, however, the same value of $\varphi + \omega t$ is obtained for more than one value of α (including also those values of α which fall outside the interval from 0 to 2π; these values can be obtained from the results for α between 0 and 2π by adding to or subtracting from $\varphi + \omega t$ a multiple of 2π), ψ becomes equal to the right-hand side of (4.12) but is summed over all values of α which lead to the same value of $\varphi + \omega t$. This situation has been explained in Fig. 3. In case I, only one α corresponds to each $\varphi + \omega t$, but in case II, the same value λ of $\varphi + \omega t$ is obtained from the three α values, α_1, α_2, and α_3. For obtaining ψ, one has to sum the right-hand side of (4.12) over α_1, α_2, and α_3' $(= \alpha_3 - 2\pi)$.

When there is more than one α value leading to the same value of $\varphi + \omega t$, it means that there is also more than one value of φ_0. This, in turn, means that at the point x,r,φ at time t there are received disturbances from more than one point of the helicoidal line along which the source moves.

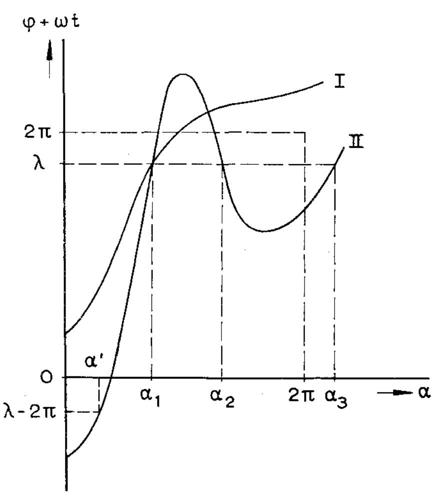

Fig. 3 Relation between $\varphi + \omega t$ and α

Since the disturbances move with the speed of sound, this is possible only if the source itself moves with a speed that is larger than or at least equal to the speed of sound. Indeed, it now will be shown mathematically that only if the total velocity $(U^2 + \omega^2 r_0^2)^{1/2}$ of the source is at least equal to the speed of sound will more than one α value yield the same value of $\varphi + \omega t$.

The right-hand side of Eq. (4.14) is a monotonically increasing function of α if its derivative to α is everywhere positive. This derivative is equal to

$$1 + \frac{\omega r\, r_0 \sin\alpha}{c[x^2 + \beta^2(r^2 + r_0^2 - 2r\, r_0 \cos\alpha)]^{1/2}} \tag{4.16}$$

It is found that the extrema of the second term occur for

$$x = 0 \qquad r \cos\alpha = r_0 \tag{4.17}$$

The derivative then becomes equal to

$$1 + (\omega r_0 \tan\alpha / \beta c \,|\tan\alpha|)$$

which will become zero if $\tan\alpha$ is negative and $\omega r_0 = \beta c$. The latter condition yields, when using the definition of β, $\omega^2 r_0^2 + U^2 = c^2$, which means that the total velocity of the source equals the speed of sound. Since (4.17) requires $\cos\alpha$ to be positive while $\tan\alpha$ is negative, it follows that $-(\pi/2) < \alpha < 0$. Figure 4 shows that the points $(0,r,\varphi)$, where the derivative of $\varphi + \omega t$ to α becomes zero, lie at the straight half-line AB. These are exactly the points where signals from two neighboring points near A of the source trajectory are received simultaneously if the total speed of the source is equal to the speed of sound. Hence, all points outside the cylinder of radius r_0 at a certain time will receive simultaneously disturbances of two neighboring points.

If the source is moving supersonically but U is still subsonic, β/ω becomes smaller, and it can be shown that disturbances of more than one point are received simultaneously at a certain time for any point outside the cylinder for which the radius r is determined by $\omega r = \beta c$. This yields $r < r_0$.

5. Potential Due to a Double Moving along a Helicoidal Line

As was shown in Sec. 3, the lifting problem is connected with a doublet distribution. Therefore, the potential due to a doublet perpendicular to the helicoidal surface $H = 0$ will be investigated now. Since the propeller must give a certain thrust, it follows from Fig. 1 that the pressure at the side H^-, where H becomes negative and φ smaller, is larger than at the other side H^+. The relation between pressure and acceleration potential is $p = -\rho\psi$, which shows that ψ is smaller at H^- than at H^+. Since a source gives a negative value of ψ, it is clear that the source is at H^- and the sink at H^+. First take the distance between source and sink equal to h and their strengths Q as finite. Retaining as coordinates for the source at time $t = 0(0,r_0,0)$, the coordinates for the sink become $-h \cos\gamma$, r_0, $(h/r_0) \sin\gamma$, where γ is the angle between the normal to the surface $H = 0$ and the x axis. One has

$$\cos\gamma = \omega r_0/(U^2 + \omega^2 r_0^2)^{1/2} \tag{5.1}$$

The value of ψ in the point (x,r,φ,t) due to the source in $(0,r_0,0)$ will be opposite to the value of ψ in $x - h\cos\gamma$, r, $\varphi + (h/r_0)\sin\gamma, t$ due to the sink in $-h\cos\gamma$, r_0, $(h/r_0)\sin\gamma$. Hence

$$\psi_{\text{source}}(x,r,\varphi,t) = -\frac{Q}{4\pi\sigma}$$

where $\sigma = s_0 + (\omega/c) r r_0 \sin\alpha$. Also,

$$\psi_{\text{sink}}\left(x - h\cos\gamma, r, \varphi + \frac{h}{r_0}\sin\gamma,\, t\right) = +\frac{Q}{4\pi\sigma}$$

Also,

$$\psi_{\text{sink}}(x,r,\varphi,t) + \left(-h\cos\gamma \frac{\partial}{\partial x} + \frac{h}{r_0}\sin\gamma \frac{\partial}{\partial\varphi}\right) \times \psi_{\text{sink}}(x,r,\varphi,t) = \frac{Q}{4\pi\sigma}$$

Hence

$$\psi_{\text{source}}(x,r,\varphi,t) + \psi_{\text{sink}}(x,r,\varphi,t) = \left(h\cos\gamma \frac{\partial}{\partial x} - \frac{h}{r_0}\sin\gamma \frac{\partial}{\partial\varphi}\right)\psi_{\text{sink}}(x,r,\varphi,t)$$

If one substitutes $\psi_{\text{sink}}(x,r,\varphi,t) = Q/4\pi\sigma$ in the right-hand side, an error of order h^2 is made. In the limit $h \to 0$ and $Qh \to E$ (E = doublet strength), the error vanishes, and it is found that

$$\psi_{\text{doublet}}(x,r,\varphi,t) = \left(\cos\gamma \frac{\partial}{\partial x} - \frac{\sin\gamma}{r_0}\frac{\partial}{\partial\varphi}\right)\frac{E}{4\pi\sigma} \tag{5.2}$$

Differentiation gives

$$\begin{aligned} \frac{\partial}{\partial x}\frac{E}{4\pi\sigma} &= -\frac{E}{4\pi\sigma^2}\left(\frac{\partial s_0}{\partial x} + \frac{\omega}{c} r r_0 \cos\alpha \frac{\partial\alpha}{\partial x}\right) \\ \frac{\partial}{\partial\varphi}\frac{E}{4\pi\sigma} &= -\frac{E}{4\pi\sigma^2}\left(\frac{\partial s_0}{\partial\varphi} + \frac{\omega}{c} r r_0 \cos\alpha \frac{\partial\alpha}{\partial\varphi}\right) \end{aligned} \tag{5.3}$$

The derivatives of s_0 and α to x and φ are obtained by differentiation of (4.13) and (4.14) with respect to these variables.

Hence

$$\frac{\partial s_0}{\partial x} = \frac{1}{s_0}\left(x + \beta^2 r r_0 \sin\alpha \frac{\partial\alpha}{\partial x}\right)$$

and

$$0 = \frac{\partial\alpha}{\partial x} - \frac{\omega}{\beta^2 c} M + \frac{\omega}{\beta^2 c}\frac{\partial s_0}{\partial x}$$

This set of equations has the solution

$$\begin{aligned} \frac{\partial s_0}{\partial x} &= \frac{1}{\sigma}\left(x + M\frac{\omega}{c} r r_0 \sin\alpha\right) \\ \frac{\partial\alpha}{\partial x} &= \frac{\omega}{\beta^2 c}\,\frac{M s_0 - x}{\sigma} \end{aligned} \tag{5.4}$$

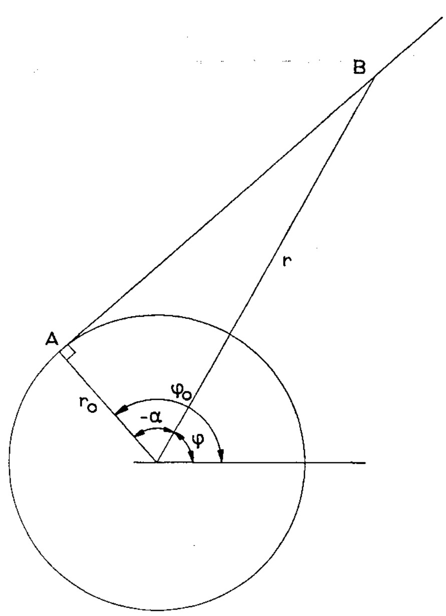

Fig. 4 Relation between various quantities when $\varphi + \omega t$ is stationary with respect to α

Also, when differentiating with respect to φ,

$$\frac{\partial s_0}{\partial \varphi} = \frac{\beta^2 r r_0 \sin\alpha}{s_0} \frac{\partial \alpha}{\partial \varphi}$$

$$1 = \frac{\partial \alpha}{\partial \varphi} + \frac{\partial s_0}{\partial \varphi} \frac{\omega}{\beta^2 c}$$

with the solution

$$\frac{\partial s_0}{\partial \varphi} = \frac{\beta^2 r r_0 \sin\alpha}{\sigma} \qquad \frac{\partial \alpha}{\partial \varphi} = \frac{s_0}{\sigma} \tag{5.5}$$

Substitution of the results (5.4) and (5.5) into Eq. (5.3) and this again in (5.2) yields

$$\psi_{\text{doublet}}(x,r,\varphi,t) = -(E/4\pi\sigma^3)[a + b(\omega/c)rr_0 \cos\alpha] \tag{5.6}$$

where

$$\sigma = s_0 + (\omega/c)rr_0 \sin\alpha$$

$$a = [x + (\omega/c)M\ rr_0 \sin\alpha] \cos\gamma - \beta^2 r \sin\alpha \sin\gamma \tag{5.7}$$

$$b = (\omega/\beta^2 c)(Ms_0 - x)\cos\gamma - (s_0/r_0)\sin\gamma$$

The quantities γ and s_0 still are given by Eqs. (5.1) and (4.13), respectively, and the relation between φ and α again by (4.14).

It is seen that the results for source and doublet both depend on the four following dimensionless parameters:

$$M \qquad \omega r_0/c \qquad x/r_0 \qquad r/r_0$$

6. Lifting Problem

It was shown in Sec. 3 that for the lifting problem the blade can be replaced mathematically by a doublet distribution. According to Table 1, the normal acceleration follows from the blade geometry and from the form of the surface $H = 0$ (the latter being determined by the ratio ω/U). In order to obtain the acceleration potential and hence the pressure difference from these data, the corresponding boundary value problem of Neumann should be solved. Moreover, the singularity at the leading edge has to be evaluated. This problem is not considered in the present paper. It is assumed here that the pressure difference, and hence the doublet distribution, is known. The problem is to find the acceleration potential or the pressure at arbitrary points in space due to this doublet distribution.

This means that, in order to obtain the total acceleration potential in a point (x,r,φ) at time t, one has to replace in Eq. (5.7) the concentrated doublet strength E by a doublet distribution of strength e per unit area and has to integrate over the blade surface. The area element of the blade surface (taken at $H = 0$ in the linearized theory) becomes equal to

$$[1 + (\omega^2 r_0^2/U^2)]^{1/2} dx_0 dr_0$$

In performing the integration in (5.7), the length x must be replaced by $x - x_0$ everywhere.

The double integration that arises in this way may be simplified by using a lifting line approach. In each chordwise strip of the blade, the pressure difference then is concentrated along the quarter-chord axis, thus avoiding the integration of (5.7) over x_0.

A second, more drastic approximation, which, however, may be allowed for many practical cases where the noise due to the lower harmonics has to be evaluated at some distance from the propeller, is to concentrate also the loads along the quarter-chord axis in one single point. According to Deming,[9] this point best can be taken at about 80% of the blade span. The strength of the doublet to be placed at this point is equal to the total lift of the blade divided by the air density ρ. The component of the lift in the forward direction is obtained by multiplication with $\cos\gamma$. If T denotes the total thrust of the propeller and n the number of blades, then the doublet strength is given by‡

$$\bar{e} = T/\rho n \cos\gamma$$

The formula for the pressure due to one blade then becomes

$$p(x,r,\psi,t) = \frac{T}{4\pi n[s_0 + (\omega/c)rr_0 \sin\alpha]^3} \left[\bar{a} + \bar{b}\left(\frac{\omega}{c}\right) rr_0 \cos\alpha\right] \tag{6.1}$$

where

$$\bar{a} = x + (\omega/c)M\ rr_0 \sin\alpha - \beta^2 r \sin\alpha \tan\gamma$$
$$\bar{b} = (\omega/\beta^2 c)(Ms_0 - x) - (s_0/r_0)\tan\gamma \tag{6.2}$$

The value of r_0 should be taken equal to 80% of the outer radius of the blade.

For the pressure due to a propeller of n blades, one has to take

$$\sum_{i=0}^{n-1} p\left(x,r,\varphi + \frac{2\pi i}{n}, t\right) \tag{6.3}$$

In Fig. 5, results are presented for the dimensionless pressure $p/(T/nr_0^2)$ as a function of time during one revolution of a one-bladed propeller when the Mach number of forward flight is 0.5, the Mach number $\omega r_0/c$ corresponding to the rotational speed of the doublet is 0.56, and the parameters x/r_0 and r/r_0 take several values.

In order to obtain the noise level, one has to proceed as follows. Perform a harmonic analysis to the pressure given by Eq. (6.1) as a function of $\varphi + \omega t$. This yields the pressure amplitude p_m for each component of a one-bladed propeller. When an n-bladed propeller is considered, only the components of which the order is a multiple of n have to be taken into account, their magnitude being n times the value for one blade. This agrees with the fact that the resulting propeller will have periodicity $2\pi/n$ in time. The noise level in decibels for each harmonic can be found with the aid of the formula

$$D_b = 20 \log_{10}(np_m/p_{\text{ref}}) \tag{6.4}$$

where $p_{\text{ref}} = 0.0002$ dynes/cm² and m is a number divisible by n.

In Fig. 6, the resulting noise level in decibels has been given for the first three harmonics of a four-bladed propeller for which T/nr_0^2 is equal to 49.08 kg/m² = 4813 dynes/cm². This corresponds to a thrust per unit disk area $T/\pi(\frac{5}{4}r_0)^2 =$ 40 kg/m².

7. Thickness Problem

According to Sec. 3, the propeller blade could be replaced in the thickness problem by a distribution of acceleration potential sources with a strength equal to

$$\epsilon\, CU^2(\partial^2/\partial x^2)(f^+ - f^-)$$

However, it appears to be simpler for the thickness problem to use a distribution of velocity potential sources. The strength of this distribution is

$$\epsilon\, CU(\partial/\partial x)(f^+ - f^-)$$

since, for an observer moving with the blade, the flow is steady and for a steady flow holds $\psi = U(\partial\chi/\partial x)$, where ψ is the acceleration potential and χ the velocity potential. Also, this strength is in accordance with the value of the discon-

‡ It is assumed here that the thrust is due completely to pressures acting normal to the blade. The reduction of the thrust due to skin friction has been neglected.

tinuity in the normal velocity component, which follows from the results of Sec. 2.

The total source strength in a chordwise section becomes

$$\epsilon\, CU \int_{xl}^{xt} \frac{\partial}{\partial x}(f^+ - f^-)\left(1 + \frac{\omega^2 r^2}{U^2}\right)^{1/2} dx = \epsilon r U \int_{xl}^{xt} \frac{\partial}{\partial x}(f^+ - f^-)dx = 0$$

The factor $[1 + (\omega^2 r^2/U^2)]^{1/2}$ has been added to the integrand, since the chord makes an angle with the x axis (see the expression for the blade element area in Sec. 6). The value of C given by Eq. (2.5) has been used in the reduction.

In first approximation, the source distribution in each chordwise section can be replaced by a single doublet, the

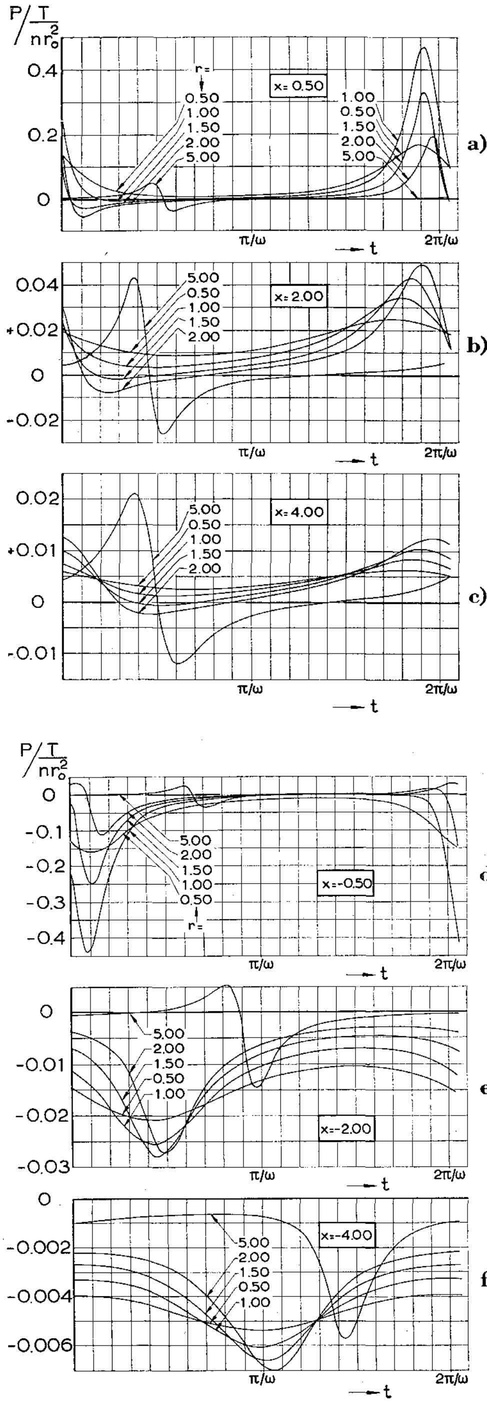

Fig. 5 Pressure due to lift as a function of time for three positive values of x and several values of r

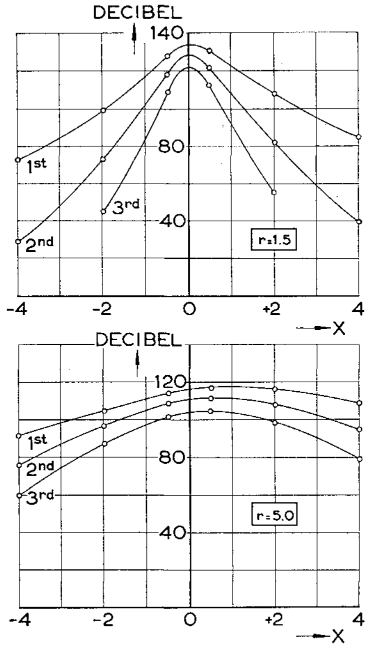

Fig. 6 Noise level in decibels as a function of x for two values of r for the first three harmonics (lift case)

strength of which is

$$\epsilon\, CU \int_{xl}^{xt} x \frac{\partial}{\partial x}(f^+ - f^-)\left(1 + \frac{\omega^2 r^2}{U^2}\right)^{1/2} dx = \epsilon r U \int_{xl}^{xt} (f^+ - f^-)dx$$

According to Eq. (2.2), $\epsilon(f^+ - f^-)$ gives the angle under which the thickness of the blade at the point (x,r) is seen from the x axis.

Hence

$$A = \epsilon r \int_{xl}^{xt} (f^+ - f^-)dx \tag{7.1}$$

is the profile area in the section r. The doublet strength is equal to AU.

In a still more drastic approximation, the whole blade thickness effect could be taken into account by a single doublet of strength VU, where V is the volume of the blade. This doublet will be taken again at a distance r_0 from the axis, although r_0 probably will be less than 80% of the blade span, as was assumed in the lift case.

The pressure due to a rotating velocity potential doublet in the true direction now will be calculated. As coordinates for the source, one takes again at time $t = 0$ $(0,r_0,0)$. The coordinates of the sink then become $h \sin\gamma$, r_0, $(h/r_0)\cos\gamma$, where γ again is given by Eq. (5.1), whereas h denotes the distance between source and sink. Comparing this configuration with that of Sec. 5, it is seen that, if one replaces γ in the coordinates of the sink of Sec. 5 by $\gamma + (\pi/2)$, the coordinates of the sink of the present section are obtained. Hence, also the resulting velocity potential is obtained from (5.6) and (5.7) by this transformation, which leads to

$$\chi_{\text{doublet}}(x,r,\varphi,t) = (E/4\pi\sigma^3)[a_1 + b_1(\omega/c)rr_0\cos\alpha] \tag{7.2}$$

where

$$\begin{aligned}\sigma &= s_0 + (\omega/c)rr_0 \sin\alpha \\ a_1 &= [x + (\omega/c)Mr\, r_0 \sin\alpha]\sin\gamma + \beta^2 r \sin\alpha\cos\gamma \qquad (7.3)\\ b_1 &= (\omega/\beta^2 c)(Ms_0 - x)\sin\gamma + (s_0/r_0)\cos\gamma\end{aligned}$$

From this expression, the pressure has to be calculated by aid of the formula

$$p(x,r,\varphi,t) = -\rho[(\partial\chi/\partial t) + U(\partial\chi/\partial x)]$$

In Eqs. (7.2) and (7.3), the quantities s_0 and α are functions of t and x. Their derivatives follow from Eqs. (4.13) and (4.14) in a similar way as in Sec. 5, where s_0 and α have been differentiated to x and φ. Equation (5.4) can be used for the derivatives to x. Also, one finds

$$\frac{\partial s_0}{\partial t} = \frac{\omega\beta^2 r r_0 \sin\alpha}{\sigma} \qquad \frac{\partial\alpha}{\partial t} = \frac{\omega s_0}{\sigma} \tag{7.4}$$

From Eq. (7.2),

$$\frac{\partial\chi}{\partial x} = -\frac{3E}{4\pi\sigma^4}\left(a_1 + b_1\frac{\omega}{c}rr_0\cos\alpha\right)\left(\frac{\partial s_0}{\partial x} + \frac{\omega}{c}rr_0\cos\alpha\frac{\partial\alpha}{\partial x}\right) + \frac{E}{4\pi\sigma^3}\left(\frac{\partial a_1}{\partial\alpha} + \frac{\omega}{c}rr_0\cos\alpha\frac{\partial b_1}{\partial x} - b_1\frac{\omega}{c}rr_0\sin\alpha\frac{\partial\alpha}{\partial x}\right)$$

where

$$\frac{\partial a_1}{\partial x} = \sin\gamma + \frac{\omega}{c}M\,rr_0\sin\gamma\cos\alpha\frac{\partial\alpha}{\partial x} + \beta^2 r\cos\alpha\cos\gamma\frac{\partial\alpha}{\partial x}$$

$$\frac{\partial b_1}{\partial x} = \frac{\omega}{\beta^2 c}M\sin\gamma\frac{\partial s_0}{\partial x} - \frac{\omega}{\beta^2 c}\sin\gamma + \frac{1}{r_0}\cos\gamma\frac{\partial s_0}{\partial x}$$

After substitution, one finds

$$\frac{\partial\chi}{\partial x} = -\frac{3E}{4\pi\sigma^5}\left(a_1 + b_1\frac{\omega}{c}rr_0\cos\alpha\right)\times\left\{x + M\frac{\omega}{c}rr_0\sin\alpha + \frac{\omega^2}{\beta^2c^2}rr_0\cos\alpha(Ms_0 - x)\right\} + \frac{E}{4\pi\sigma^3}\left\{\left(1 - \frac{\omega^2}{c^2}rr_0\cos\alpha\right)\sin\gamma + M\frac{\omega}{c}r\cos\alpha\cos\gamma\right\} - \frac{E}{4\pi\sigma^4}\frac{\omega^2}{\beta^2c^2}rr_0(Ms_0 - x)b_1\sin\alpha \tag{7.5}$$

Similarly, one obtains

$$\frac{\partial\chi}{\partial t} = -\frac{3E}{4\pi\sigma^5}\left(a_1 + b_1\frac{\omega}{c}rr_0\cos\alpha\right)\times\left(\beta^2\sin\alpha + \frac{\omega}{c}s_0\cos\alpha\right)\omega rr_0 + \frac{E}{4\pi\sigma^3}\left(M\frac{\omega^2}{c}rr_0\sin\gamma\cos\alpha + \beta^2\omega r\cos\gamma\cos\alpha\right) - \frac{E}{4\pi\sigma^4}\frac{\omega^2}{c}rr_0s_0b_1\sin\alpha \tag{7.6}$$

Finally, the pressure becomes equal to

$$p(x,r,\varphi,t) = \frac{3E\rho c}{4\pi\sigma^5}\left(a_1 + b_1\frac{\omega}{c}rr_0\cos\alpha\right)\times\left(1 - \frac{\omega^2}{\beta^2c^2}rr_0\cos\alpha\right)(s_0 - Mx) - \frac{3E\rho c}{4\pi\sigma^4}\left(a_1 + b_1\frac{\omega}{c}rr_0\cos\alpha\right) - \frac{E\rho c}{4\pi\sigma^4}\frac{\omega^2}{\beta^2c^2}rr_0(s_0 - Mx)b_1\times\sin\alpha + \frac{E\rho c}{4\pi\sigma^3}\left(M\sin\gamma + \frac{\omega}{c}r\cos\gamma\cos\alpha\right) \tag{7.7}$$

This is the pressure due to the thickness of one blade. In Fig. 7, results have been presented for the dimensionless pressure $p/(\rho U^2V/r_0^3)$ as a function of time during a revolution of a one-bladed propeller. Here $V = E/U$ denotes the volume of the blade. The Mach number of forward flight again has been taken equal to 0.5 and that of the rotational speed equal to 0.56.

Similarly as for the lifting case, the noise level also has been calculated for the pressure fluctuation due to thickness. For the four-bladed propeller considered, $V/r_0^3 = 0.004$ has been assumed for each blade, where r_0 again has been taken equal to 80% of the outer radius of the blade. The quantity $\rho U^2V/r_0^3$ takes, then, the value of 1353 dynes/cm² at sea level. The noise level, which has been calculated by aid of Eq. (6.4), is given in Fig. 8 as a function of x/r_0 for two different values of r/r_0.

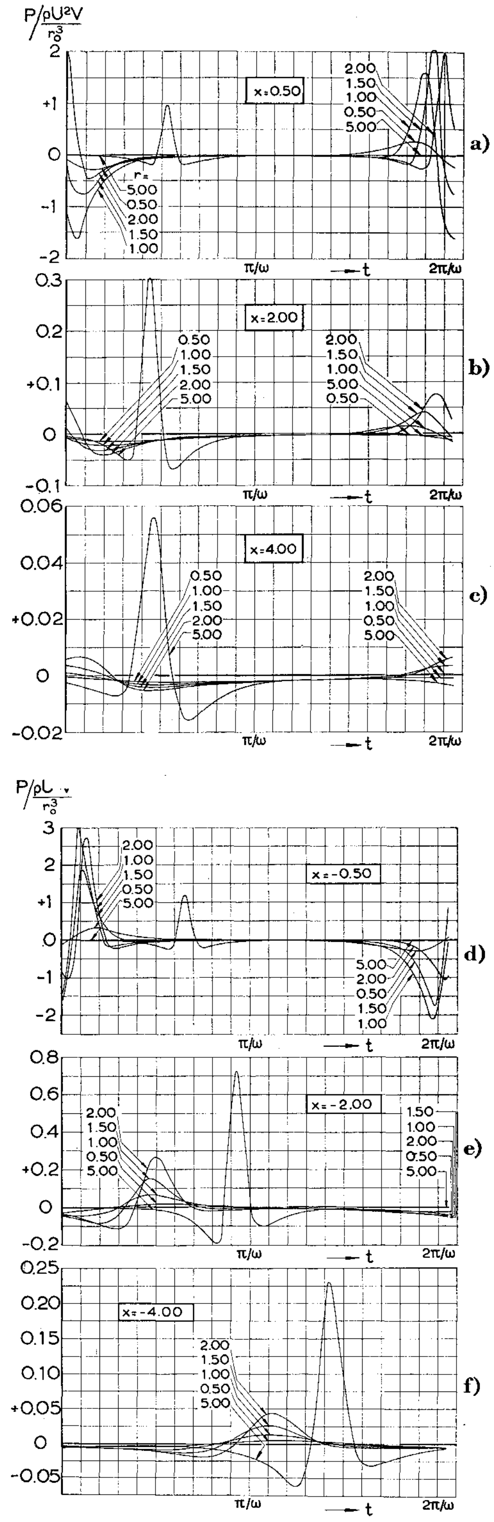

Fig. 7 Pressure due to thickness as a function of time for three positive values of x and several values of r

8. Final Considerations

Since for the lifting case and the thickness case the value of r_0 has been taken equal and since moreover the noise levels have been calculated for the same ratios x/r_0 and r/r_0 in both cases, it follows that the noise levels are comparable directly as they refer to the same points in space. It is seen from the results (Figs. 6 and 8) that the noise due to thickness is of

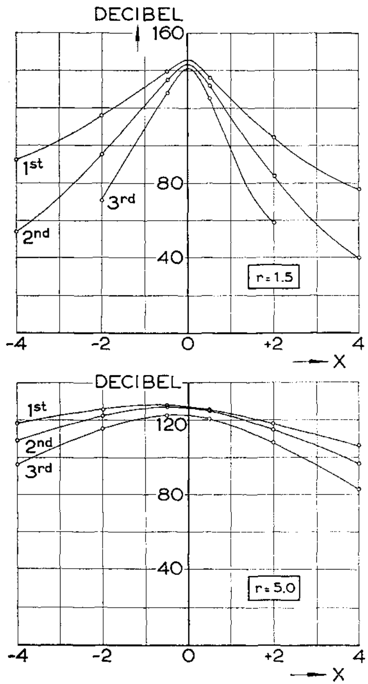

Fig. 8 Noise level in decibels as a function of x for two values of r for the first three harmonics (thickness case)

the same order of magnitude as that due to thrust and torque. This is in agreement with the results obtained by Arnoldi.[3]

The simplification of the source and doublet distributions to one single doublet in each case is of course a rather rough simplification and may, in fact, influence the numerical results to a certain extent, especially at small distances from the blade. It may be remarked here that a calculation taking into account a larger number of doublets or sources for the representation of the distributions also, in principle, can be made fairly easy. The complication is that for the various doublets or sources α must be taken in such a way that this leads to contributions in $\psi(x,r,\varphi,t)$ or $\chi(x,r,\varphi,t)$, for which $\varphi + \omega t$ always takes the same value. This can be obtained by aid of a few iteration steps. A certain value is taken for $\varphi + \omega t$, and a trial value for α is assumed. From Eq. (4.13), s_0 is found, and, by substituting this value in Eq. (4.14), an improved value of α is obtained. This procedure can be repeated until convergence. Moreover, one has to take into account that the doublets or sources have different positions (x_0,r_0,φ_0) at time $t = 0$.

Using such a concept of a surface distribution of sources and doublets, the investigations presented here enable the calculation of the pressure field around a rotating propeller with the same order of accuracy as is obtained in ordinary lifting surface theory. These calculations are relatively simple due to the use of a parameter representation for the singularity moving along a helical path.

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